Multivariate Normal Distribution

Univariate normal distribution

P.d.f $\quad f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \quad ; -\infty < x < \infty$

where, μ and σ are Population Parameters

Note: x - univariate

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[(x-\mu)^T \Sigma^{-1} (x-\mu)\right]}$$

Multivariate normal distribution

Univariate x, ~~...~~ μ and σ are Converted to

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} \qquad \text{Mean vector } \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}$$

Note:

For univariate σ^2

and for multivariate Σ^{-1}

Multivariate Covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & & & \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix}_{p\times p} \quad (\text{or}) \quad \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \vdots & & & \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_p^2 \end{bmatrix}$$

$$f(x) \sim N_p(\mu, \Sigma)$$

Bivariate normal distribution

let x_1 and x_2 be two variables

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$$

Assume, x_1 and x_2 are independent

then $\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$

$$\boxed{|\Sigma| = (\sigma_1\sigma_2)^2} \longrightarrow (1)$$

Joint density function of x_1 and x_2 is the product of marginal density function of x_1 and x_2

$$f(x_1, x_2) = f(x_1) \cdot f(x_2)$$

$$= \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} \cdot \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}$$

$-\infty < x_1 < \infty$
$-\infty < x_2 < \infty$

$$= \frac{1}{\sigma_1\sigma_2 (2\pi)^{2/2}} e^{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}$$

$$= \frac{1}{|\Sigma|^{1/2} (2\pi)^{2/2}} e^{-\frac{1}{2}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]}$$

$$f(x_1, x_2) = \frac{1}{|\Sigma|^{1/2} (2\pi)^{2/2}} e^{-\frac{1}{2}\left[(x-\mu)^T \Sigma^{-1} (x-\mu)\right]}$$

In general;

$$f(x_1, x_2 \dots x_p) = \frac{1}{|\Sigma|^{1/2} (2\pi)^{p/2}} e^{-\frac{1}{2}\left[(x-\mu)^T \Sigma^{-1} (x-\mu)\right]}$$

$-\infty < x_j < \infty$; $j = 1, 2, \dots p$

This is multivariate normal distribution ~~density~~ function. density.

Note:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \Rightarrow \Sigma^T = \begin{bmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix}$$

$$\Sigma^{-1} = \frac{1}{|\Sigma|} (\text{adj } \Sigma)$$

$$\boxed{\Sigma^{-1} = \frac{1}{\sigma_1^2 \sigma_2^2} \begin{bmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix}}$$

Now, $-\frac{1}{2} \begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix}$

PDF of Multivariate Normal Distribution

Proof:

We know that the p.d.f of univariate normal distribution is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \qquad -\infty < x < \infty$$

Let the mean $\mu = \beta$ and the variance $\sigma^2 = \frac{1}{\alpha}$ and $k = \frac{1}{\sigma\sqrt{2\pi}}$

then $f_X(x) = k\, e^{-\frac{1}{2}\alpha(x-\beta)^2}$

$$f_X(x) = k\, e^{-\frac{1}{2}\alpha(x-\beta)(x-\beta)} \longrightarrow (1)$$

The density function of a multivariate normal distribution of $X_1, X_2, \cdots X_p$ has an analogous form.

The Scalar variable 'x' in equation (1) is replaced by a vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} \longrightarrow (2)$$

and the Scalar Constant β, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix} \longrightarrow (3)$

and the positive Constant α,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & & a_{pp} \end{bmatrix} \longrightarrow (4)$$

The square term $\alpha(x-\beta)^2 = (x-\beta)\alpha(x-\beta)$ is replaced by the quadratic form

$$(x-b)'A(x-b) = \sum_{i,j=1}^{p} a_{ij}(x_i - b_i)(x_j - b_j) \longrightarrow (5)$$

Thus the density function of p variate normal distribution is

$$f(x_1, x_2 \ldots x_p) = k\, e^{-\frac{1}{2}(x-b)'A(x-b)} \quad ; \quad k > 0 \longrightarrow (6)$$

Since A is positive definite

$(x-b)' A (x-b) \geq 0$ → (a)

$f(x_1 \dots x_p) \leq k$ → (b)

Now, to determine K such that the integral of equation (6) over 'p' dimensional space is unity

Let $\frac{1}{K} = K^* = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-b)' A (x-b)} \, dx_p \dots dx_2 \, dx_1$ → (7)

Since A is a positive definite, there exist a non singular matrix C such that

$C' A C = I$ → (8)

Where I, identity matrix and C' is the transpose of C

Let $x - b = C y$

where $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}$

Then $(x-b)' A (x-b) = (CY)' A (CY)$

$= (y'C') A (CY)$

$= Y' I Y$

$$\boxed{(x-b)' A (x-b) = Y'Y} \longrightarrow (9)$$

The Jacobian transformation is

$X - b = CY$

$X = CY + b$

$J = \frac{\partial x}{\partial y} = C$

$$\boxed{|J| = \left|\frac{\partial x}{\partial y}\right| = |C|}$$

Note: $|C|$ - absolute value

Substitute (9) in ~~(8)~~ (7)

(7) $\Rightarrow$ $K^* = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}y'y} \, |C| \, dy_p \cdots dy_2 \, dy_1 \longrightarrow$ (10)

But $y'y = (y_1 \; y_2 \; \cdots \; y_p)\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}$

$= y_1^2 + y_2^2 + \cdots + y_p^2$

$$\boxed{y'y = \sum_{i=1}^{p} y_i^2}$$

(10) $\Rightarrow$ $K^* = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum_{i=1}^{p} y_i^2} \, |C| \, dy_p \cdots dy_2 \, dy_1$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_{i=1}^{p} e^{-\frac{1}{2}y_i^2} \, |C| \, dy_p \cdots dy_2 \, dy_1$

$= |C| \left\{ \prod_{i=1}^{p} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y_i^2} \, dy_i \right\}$

$= |C| \prod_{i=1}^{p} \left\{ \sqrt{2\pi} \right\}$ {$\because$ By using p.d.f of standard normal distribution over $(-\infty, \infty)$)

~~$= |C| (2\pi)^{\frac{p}{2}}$~~

$= |C| \left\{ \sqrt{2\pi} \cdot \sqrt{2\pi} \cdot \cdots \; p \text{ times} \right\}$

$= |C| \left(\sqrt{2\pi}\right)^p$

$= |C| (2\pi)^{p/2}$ {$\because$ From (8)

$C'AC = I$

$|C'AC| = |I|$

$|C'| \, |A| \, |C| = 1$

$|C|^2 |A| = 1$

$|C| = \frac{1}{\sqrt{|A|}}$

$K^* = \frac{1}{\sqrt{|A|}} (2\pi)^{p/2}$

$$\boxed{\frac{1}{K^*} = \sqrt{|A|} \, (2\pi)^{-p/2}}$$

7

$\Rightarrow K = \sqrt{|A|} \, (2\pi)^{-p/2}$

$\therefore$ The multivariate normal density function is,

$$f(x_1 x_2 \dots x_p) = \frac{|\sqrt{A}|}{(2\pi)^{p/2}} e^{-\frac{1}{2}(x-b)' A (x-b)} \longrightarrow (11)$$

Now to determine b and A, Find the first two moments of $x_1\ x_2 \dots x_p$

let $\quad X - b = CY$

$$X = CY + b$$

$$E(X) = C\,E(Y) + b \longrightarrow (12)$$

But $\quad E(Y) = 0, \quad E(YY') = I$

$$\therefore E(X) = b = \mu \longrightarrow (13)$$

Covariance matrix of X, $\Sigma = E\left[(X-\mu)(X-\mu)'\right]$

$$= E\left[(CY+b-\mu)(CY+b-\mu)'\right]$$

$$= E\left[(CY)(CY)'\right]$$

$$= E\left[(CY)(Y'C')\right]$$

$$= C\,E(YY')\,C'$$

$$= CIC'$$

$$\boxed{\Sigma = CC'} \longrightarrow (14)$$

$(8) \Rightarrow \quad C'AC = I$

$$AC = (C')^{-1} I$$

$$A = (C')^{-1} I C^{-1}$$

$$= (C')^{-1} C^{-1}$$

$$A = (CC')^{-1}$$

$$\boxed{A^{-1} = CC'}$$

$(14) \Rightarrow \quad \boxed{\Sigma = A^{-1}}$

$\Rightarrow \quad A\Sigma = I$

$|A||\Sigma| = 1$

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$$|A| = \frac{1}{|\Sigma|}$$

$$|\sqrt{A}| = \frac{1}{|\Sigma|^{1/2}}$$

$\therefore$ The p.d.f of multivariate normal distribution is

(11) $\Rightarrow$ $$f(x_1, x_2 \dots x_p) = \frac{|\sqrt{A}|}{(2\pi)^{p/2}} e^{-\frac{1}{2}(x-\mu)'(\Sigma^{-1})(x-\mu)}$$

$$= \frac{1}{(2\pi)^{p/2}} \cdot \frac{1}{|\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

Symbolically $X \sim N_p(\mu, \Sigma)$

—x—

Properties of multivariate normal distribution

Property ① Moment generating function (MGF) of multivariate normal distribution.

Let $X \sim N_p(\mu, \Sigma)$ then

$$M_X(t) = e^{t'X + \frac{1}{2}t'\Sigma t}$$

Proof

$M_X(t) = E(e^{t'X})$ where $t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_p \end{pmatrix}$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{t'x} f_X(x)\, dx$$

$$= \int\cdots\int e^{t'x} \frac{1}{(2\pi)^{p/2}|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)} dx_p \cdots dx_2\, dx_1$$

$$= \int\cdots\int \frac{1}{(2\pi)^{p/2}|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu) + t'x} dx_p \cdots dx_1 \quad \longrightarrow ①$$

Consider,

$$e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu) + t'x} = \cancel{\exp\left\{-\frac{1}{2}(x-\mu-\Sigma t+\Sigma t)'\Sigma^{-1}(x-\mu-\Sigma t+\Sigma t)\right\}}$$

$$= \exp\left\{-\frac{1}{2}(x-\mu-\Sigma t+\Sigma t)'\Sigma^{-1}(x-\mu-\Sigma t+\Sigma t) + t'x\right\}$$

$$= \exp\left\{-\frac{1}{2}(x-\mu-\Sigma t)'\Sigma^{-1}(x-\mu-\Sigma t) - \frac{1}{2}(\Sigma t)'\Sigma^{-1}(x-\mu-\Sigma t) - \frac{1}{2}(x-\mu-\Sigma t)'\Sigma^{-1}\Sigma t - \frac{1}{2}(\Sigma t)'\Sigma^{-1}(\Sigma t) + t'x\right\}$$

$$= \exp\left\{-\frac{1}{2}(x-\mu-\Sigma t)'\Sigma^{-1}(x-\mu-\Sigma t) - \frac{1}{2}t'\Sigma'\Sigma^{-1}(x-\mu-\Sigma t) - \frac{1}{2}(x-\mu-\Sigma t)'t - \frac{1}{2}t'\Sigma'\Sigma^{-1}(\Sigma t) + t'x\right\}$$

$$= \exp\left\{-\frac{1}{2}(x-\mu-\Sigma t)'\Sigma^{-1}(x-\mu-\Sigma t) - \frac{1}{2}t'\Sigma'\Sigma^{-1}(x-\mu) + \frac{1}{2}t'\cancel{\Sigma'\Sigma^{-1}}(\Sigma t) - \frac{1}{2}(x-\mu-\Sigma t)'t - \frac{1}{2}t'\cancel{\Sigma'\Sigma^{-1}}(\Sigma t) + t'x\right\}$$

$$= \cancel{\exp\left\{-\frac{1}{2}(x-\mu)\right.}$$

$$= \exp\left\{-\frac{1}{2}(x-\mu-\Sigma t)'\Sigma^{-1}(x-\mu-\Sigma t) - \frac{1}{2}t'(x-\mu)\right.$$
$$\left. \cancel{-\frac{1}{2}(x-\mu)} - \frac{1}{2}(x-\mu-\Sigma t)'t + t'x\right\}$$

$$= \exp\left\{-\frac{1}{2}(x-\mu-\Sigma t)'\Sigma^{-1}(x-\mu-\Sigma t) - \frac{1}{2}t'x + \frac{1}{2}t'\mu\right.$$
$$\left. - \frac{1}{2}x't + \frac{1}{2}\mu't + \frac{1}{2}(\Sigma't)t + t'x\right\}$$

$$= \exp\left\{-\frac{1}{2}(x-\mu-\Sigma t)'\Sigma^{-1}(x-\mu-\Sigma t) + t'\mu + \frac{1}{2}t'(\Sigma t)\right\} \quad \to (2)$$

Sub (2) in (1)

Now,

$$(1) \Rightarrow M_X(t) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}}\int\cdots\int \exp\left\{-\frac{1}{2}(x-\mu-\Sigma t)'\Sigma^{-1}(x-\mu-\Sigma t) + t'\mu + \frac{1}{2}t'(\Sigma t)\right\} dx_p \ldots dx_1$$

$$= \frac{\exp\left(t'\mu + \frac{1}{2}t'\Sigma t\right)}{(2\pi)^{p/2}|\Sigma|^{1/2}}\int\cdots\int \exp\left\{-\frac{1}{2}(x-\mu-\Sigma t)'\Sigma^{-1}(x-\mu-\Sigma t)\right\} dx_p \ldots dx_1$$

By Transformation,

let $Y = x-\mu-\Sigma t$

$dy = dx$

$$= \frac{\exp\left(t'\mu + \frac{1}{2}t'\Sigma t\right)}{(2\pi)^{p/2}|\Sigma|^{\frac{1}{2}}}\int\cdots\int \exp\left\{-\frac{1}{2}Y'\Sigma^{-1}Y\right\} dy_p \ldots dy_1$$

$$= \frac{\exp\left\{t'\mu + \frac{1}{2}t'\Sigma t\right\}}{(2\pi)^{p/2}|\Sigma|^{\frac{1}{2}}} (2\pi)^{p/2} (\Sigma)^{\frac{1}{2}}$$

$$\boxed{M_X(t) = \exp\left\{t'\mu + \frac{1}{2}t'\Sigma t\right\}}$$

which is the m.g.f. multivariate normal distribution.

Property ② characteristic function of multivariate normal distribution:

Statement: Let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}$ be normally distributed with mean vector μ and variance covariance matrix Σ.

The characteristic function of the random variable X is

$$\phi_X(t) = e^{it'\mu - \frac{1}{2}t'\Sigma t} \quad \text{for every real vector 't'.}$$

Proof: By the defn. of characteristic function,

$$\phi_X(t) = E\left[e^{it'x}\right]$$

$$\phi_X(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{it'x} f_{x_1 x_2 \ldots x_p}\, dx_p \ldots dx_2\, dx_1$$

$$= \int\int\cdots\int e^{it'x} \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)} dx_p \ldots dx_1 \quad \to ①$$

$$= \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \int\cdots\int e^{it'x} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)} dx_p \ldots dx_1$$

Consider $x-\mu = CY$

\# $X = CY + \mu$ | when C – is a nonsingular matrix of order $p \times p$ and $C'\Sigma^{-1}C = I$

$$\frac{\partial x}{\partial y} = C$$

$$\boxed{|J| = \left|\frac{\partial x}{\partial y}\right| = |C|} \to ②$$

$$① \Rightarrow \phi_X(t) = \frac{|C|}{(2\pi)^{p/2} |\Sigma|^{1/2}} \int \cdots \int e^{it'(CY+\mu)} e^{-\frac{1}{2}(CY)' \Sigma^{-1}(CY)} dy_p \cdots dy_1$$

$$= \frac{|C| e^{it'\mu}}{(2\pi)^{p/2} |\Sigma|^{1/2}} \int \cdots \int e^{it'CY} e^{-\frac{1}{2}(Y'C'\Sigma^{-1}CY)} dy_p \cdots dy_1$$

$$= \frac{|C| e^{it'\mu}}{(2\pi)^{p/2} |\Sigma|^{1/2}} \int \cdots \int e^{it'CY} e^{-\frac{1}{2} Y'Y} dy_p \cdots dy_1 \longrightarrow ③$$

$$t'CY = \underset{(1\times p)}{(t_1\ t_2\ t_3 \cdots t_p)} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{p\times p} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}_{p\times 1}$$

$$t'CY = (1\times 1)$$

Take $t'C = K' = (k_1\ k_2 \cdots k_p)$

then $t'CY = K'Y$

$$= (k_1\ k_2 \cdots k_p) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}$$

$$= k_1 y_1 + k_2 y_2 + \cdots + k_p y_p$$

$$\boxed{t'CY = \sum_{j=1}^{p} k_j y_j}$$

and $Y'Y = (y_1\ y_2 \cdots y_p) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}$

$$= y_1^2 + y_2^2 + \cdots + y_p^2$$

$$\boxed{Y'Y = \sum_{j=1}^{p} y_j^2}$$

③ $\Rightarrow$

$$\phi_x(t) = \frac{|C| \, e^{it'\mu}}{(2\pi)^{p/2} |\Sigma|^{1/2}} \int \cdots \int e^{i\sum_{j=1}^{p} k_j y_j} \, e^{-\frac{1}{2}\sum_{j=1}^{p} y_j^2} \, dy_p \cdots dy_1$$

$$= \frac{|C| \, e^{it'\mu}}{(2\pi)^{p/2} |\Sigma|^{1/2}} \int \cdots \int e^{i\sum_{j=1}^{p} k_j y_j - \frac{1}{2}\sum_{j=1}^{p} y_j^2} \, dy_p \cdots dy_1$$

$$= \frac{|C| \, e^{it'\mu}}{(2\pi)^{p/2} |\Sigma|^{1/2}} \prod_{j=1}^{p} \int e^{-\frac{1}{2}\left[y_j^2 - 2ik_j y_j\right]} dy_j$$

$$= \frac{|C| \, e^{it'\mu}}{(2\pi)^{p/2} |\Sigma|^{1/2}} \prod_{j=1}^{p} \int e^{-\frac{1}{2}\left[y_j^2 - 2ik_j y_j + (ik_j)^2 - (ik_j)^2\right]} dy_j$$

$$= \frac{|C| \, e^{it'\mu}}{(2\pi)^{p/2} |\Sigma|^{1/2}} \prod_{j=1}^{p} \int e^{-\frac{1}{2}\left[(y_j - ik_j)^2 + k_j^2\right]} dy_j$$

$$= \frac{|C| \, e^{it'\mu}}{(2\pi)^{p/2} |\Sigma|^{1/2}} \prod_{j=1}^{p} e^{-\frac{1}{2}k_j^2} \int e^{-\frac{1}{2}(y_j - ik_j)^2} dy_j$$

$$= \frac{|C| \, e^{it'\mu}}{(2\pi)^{p/2} |\Sigma|^{1/2}} \prod_{j=1}^{p} e^{-\frac{1}{2}k_j^2} \left(\sqrt{2\pi} \; \sqrt{2\pi} \; \ldots \; p \text{ times}\right)$$

$$= \frac{|C| \, e^{it'\mu}}{(2\pi)^{p/2} |\Sigma|^{\frac{1}{2}}} \prod_{j=1}^{p} e^{-\frac{1}{2}k_j^2} (2\pi)^{p/2}$$

$$= e^{it'\mu} \prod_{j=1}^{p} e^{-\frac{1}{2}k_j^2}$$

$\{\because \; C' \Sigma^{-1} C = I$

$|C' \Sigma^{-1} C| = 1$

$\Rightarrow |C| = |\Sigma|^{1/2}$

$$= e^{it'\mu} \; e^{-\frac{1}{2}k'k}$$

$$\boxed{\phi_X(t) = e^{it'\mu} \; e^{-\frac{1}{2}t'\Sigma t}}$$

$$\cancel{= e^{it'\mu} \; e^{-\frac{1}{2}c'\Sigma^{-1}c}}$$

$$\cancel{= e^{it'\mu}} \; e^{-\frac{1}{2}}$$

$\{\because k' = t'c$

$k'k = (t'c)(t'c)'$

$= t'c \; c't$

$\boxed{k'k = I}$

$c' \; \Sigma^{-1}c = I$

$\Sigma^{-1}c = (c')^{-1} I$

$\Sigma^{-1}c = (c')^{-1}$

$\Sigma^{-1} = (c')^{-1}(c^{-1})$

$\Sigma^{-1} = (cc')^{-1}$

$\boxed{\Sigma = cc'}$

$k'k = (t'c)(t'c)'$

$= (t'c)(c't)$

$= t' cc' t$

$\boxed{k'k = t'\Sigma t}$

Property (3)

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$. If the covariance matrix Σ of a normal random vector X is a diagonal matrix then the component of X are all independently distributed normal random variables.

Proof:

Let $\Sigma = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & \sigma_p^2 \end{bmatrix} \Rightarrow |\Sigma| = \sigma_1^2 \cdot \sigma_2^2 \cdots \sigma_p^2$

$$\boxed{|\Sigma| = \prod_{i=1}^{p} \sigma_i^2}$$

then $|\Sigma| = \prod_{i=1}^{p} \sigma_i^2$

and $\Sigma^{-1} = \frac{1}{|\Sigma|}(\text{adj}' \, \Sigma)$

$$= \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2^2} & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_p^2} \end{bmatrix} \Rightarrow |\Sigma^{-1}| = \frac{1}{\sigma_1^2} \frac{1}{\sigma_2^2} \cdots \frac{1}{\sigma_p^2}$$

$$|\Sigma^{-1}| = \prod_{i=1}^{p} \frac{1}{\sigma_i^2}$$

But $(X-\mu)' \Sigma^{-1} (X-\mu) = \sum_{i=1}^{p} \left(\frac{x_i - \mu_i}{\sigma_i} \right)^2$

$\therefore$ The p.d.f of X is

$$f_X(x) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left\{ -\frac{1}{2} \Sigma \left(\frac{x_i - \mu_i}{\sigma_i} \right)^2 \right\} \longrightarrow (1)$$

$$\Rightarrow f_X(x) = \frac{1}{(2\pi)^{p/2}} \left\{|\Sigma|^{-1}\right\} \exp\left\{-\frac{1}{2}\sum\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2\right\}$$

$$= \frac{1}{(2\pi)^{p/2}} \left\{\frac{1}{\sigma_1^2\,\sigma_2^2 \cdots \sigma_p^2}\right\}^{1/2} \exp\left\{-\frac{1}{2}\sum\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2\right\}$$

$$= \frac{1}{(2\pi)^{p/2}} \frac{1}{\sigma_1^2\,\sigma_2^2 \cdots \sigma_p^2} \exp\left\{-\frac{1}{2}\sum\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2\right\}$$

$$f_X(x) = \prod_{i=1}^{p} \frac{1}{(2\pi)^{1/2}\sigma_i} \exp\left\{-\frac{1}{2}\sum\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2\right\}$$

$\therefore$ The Components of X are independently distributed normal random variables with mean μ_i and σ_i^2

Property ④ The distribution of linear Combination of normally distributed r.v is also normal.

ie, If $X \sim N_p(\mu, \Sigma)$ then $Y = CX \sim N_p(C\mu, C\Sigma C')$

Proof

If $X \sim N_p(\mu, \Sigma)$ ~~then~~

then $\phi_X(t) = E\left[e^{it'X}\right]$

$$\phi_X(t) = e^{\left[it'\mu - \frac{1}{2}t'\Sigma t\right]} \rightarrow ①$$

Let $Y = CX$, then $\phi_Y(t) = E\left[e^{it'Y}\right]$

$$= E\left[e^{it'CX}\right]$$

$$= E\left[e^{i(C't)'X}\right]$$

$$= e^{\left[i(C't)'\mu - \frac{1}{2}(C't)'\Sigma(C't)\right]}$$

$$\phi_Y(t) = e^{\left[it'C\mu - \frac{1}{2}t'C\Sigma C't\right]}$$

$\Rightarrow Y = CX \sim N_p(C\mu, C\Sigma C')$ {∵ from ①

Hence the distribution of linear Combination of normally distributed r.v is also normal.